Qualifying Exam in Algebra, August 2014

1. Let p be a prime number and G a finite group.

a. State the definition of a p-Sylow subgroup of G.

b. Let P be a p-Sylow subgroup of G, and let H be a subgroup of G that contains P. Assume that P is normal in H, and H is normal in G. Prove that P is normal in G.

2. Let G be a group. Let $\mathrm{Aut}(G) := \{f : G \to G \,|\, f \text{ is a group isomorphism}\}$ and let $\mathrm{Inn}(G) := \{\phi_g : G \to G \,|\, \phi_g(x) = gxg^{-1},\ g \in G\}$. It is known (you don't have to prove) that $\mathrm{Aut}(G)$ is a group with operation given by composition of functions.

a. Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

b. Let $G = S_3$ (the group of permutations of three letters). Prove that $|\mathrm{Inn}(S_3)| = 6$.

3. Let G be a finite group and let $H \subset G$ be a proper subgroup ($H \neq G$).

a. It is known that for all $g \in G$, the set $gHg^{-1} := \{ghg^{-1} \,|\, h \in H\}$ is a subgroup of G. Prove that for all $g \in G$, gHg^{-1} is isomorphic to H .

b. Prove that the number of distinct sets of the form gHg^{-1} when g ranges through the elements of G is less than or equal to the index of H in G.

c. Prove that $G \neq \bigcup_{g \in G} gHg^{-1}$.

4. Let $p \geq 3$ be a prime number, and let S_p be the group of permutations of p letters. Prove that S_p does not have any Abelian subgroups of order $p(p-1)$. (Hint: use the structure theorem of finite Abelian groups).

5. Let S be a commutative ring, and let R be a PID (principal ideal domain). Assume that $f : R \to S$ is a surjective ring homomorphism. Prove that any ideal in S is a principal ideal.

6. Let R be a commutative ring.

a. State the definition of a prime ideal of R.

b. Let I be an ideal of R. Prove that I is a prime ideal if and only if R/I is a domain.

c. Let $P, Q \subset R$ be prime ideals. Prove that

$$\mathrm{Hom}_R(R/P, R/Q) \neq 0 \Leftrightarrow P \subseteq Q.$$

7. Let R denote the subring of $\mathbf{Q}$ that consists of fractions a/b with b not divisible by 3 (this is a subring of $\mathbf{Q}$, you are not required to check this fact).

1. For each of the following subsets of R, decide whether the subset is an ideal of R or not. Give a brief justification for each answer.

a. $\{\frac{3a}{b} \,|\, b \text{ relatively prime to } 3\}$

b. $\{\ldots, -3^3, -3^2, -3, 1, 3, 3^2, 3^3, 3^4, \ldots\}$

c. $\{\frac{5a}{b} \,|\, a, b \in \mathbf{Z}, b \text{ relatively prime to } 3\}$.

d. $\{\frac{9a}{b} \,|\, a, b \in \mathbf{Z}, b \text{ relatively prime to } 3\}$.

e. $\{\frac{3a}{b} \,|\, a, b \in \mathbf{Z}, b \text{ relatively prime to } 15\}$.

2. Describe the units of R.

3. Which of the ideals from part 1. are prime ideals? Justify your answers.

4. Is R a unique factorization domain? Justify.

8. Find the minimal polynomial of $\sqrt{2} + \sqrt{5}$ over $\mathbf{Q}$ (and prove that it is indeed the minimal polynomial).

9. Let $\psi = e^{\frac{2\pi i}{8}} \in \mathbf{C}$ be a primitive 8-th root of unity.

a. State the definition of a normal extension, a separable extension, and a Galois extension of fields.

b. Prove that $L = \mathbf{Q}(\psi)$ is a Galois extension of $K := \mathbf{Q}$.

c. State the definition of the Galois group $\mathrm{Gal}(L/K)$ of a Galois extension of fields L/K.

d. Compute the Galois group $\mathrm{Gal}(L/K)$ for $L = \mathbf{Q}(\psi)$ and $K = \mathbf{Q}$.

e. For the field extension in part d., list the subgroups of $G := \mathrm{Gal}(L/K)$ and find the subgroup(s) H of G with the property that $L^H = \mathbf{Q}(i)$. (L^H is the subfield of L consisting of elements fixed by H.)

10. Let p be a prime number and n a positive integer. Let K be a finite field with p^n elements.

a. Prove that every element of K has a p-th root in K (i.e. for every $x \in K$ there exists a $y \in K$ such that $y^p = x$).

b. Show that the p-th root of any given element $x \in K$ is unique.