Qualifying Exam in Algebra – University of South Carolina, Fall 2018

Instructions: Answer all questions, closed book/notes.

Questions without multiple parts are worth 10 points; questions with multiple parts are worth 6 points for each part.

1. Prove or disprove: S_5 and D_{60} are isomorphic groups.

   (Here D_{60} is the dihedral group with 120 elements.)

2. Let G be a group of order 12 without any normal 3-Sylow subgroups. Let X be the set of 3-Sylow subgroups of G.

   Determine a group action of G on X for which the induced homomorphism $G \mapsto \mathrm{Sym}(X)$ is injective and has image A_4. Conclude that $G \cong A_4$.

3. Let $R := \mathbb{C}[x, y]$, and let $I \subseteq R$ be the subset of polynomials f which can be written in the form

$$f = c_1 x^{a_1} y^{b_1} + c_2 x^{a_2} y^{b_2} + \cdots + c_k x^{a_k} y^{b_k}$$

   for some nonnegative integer k and complex numbers c_i, and where the a_i and b_i are nonnegative integers with $a_i + b_i \geq 3$ for each i.

   (a) Prove that I is an ideal of R.

   (b) Exhibit a minimal set of generators for I.

   (c) Determine the dimension of R/I as a $\mathbb{C}$-vector space.

   (d) Is R/I a principal ideal domain? Prove or disprove.

4. Let $V := \mathrm{M}_n(\mathbb{C})$ the set of $n \times n$ matrices over $\mathbb{C}$. Let $A \in \mathrm{GL}_n(\mathbb{C})$. Determine the eigenvalues of the linear map

$$\begin{aligned} C_A : V &\to V \\ M &\mapsto AMA^{-1}. \end{aligned}$$

5. Let p be an odd prime. Prove, in the polynomial ring $\mathbb{F}_p[x]$, that we have the factorization

$$x^{p-1} - 1 = \prod_{0 \neq \alpha \in \mathbb{F}_p} (x - \alpha).$$

   Conclude *Wilson's theorem*: for each odd prime p, we have

$$(p-1)! \equiv -1 \pmod{p}.$$

6. Let $\beta = \sqrt[5]{2}$ and let $F = \mathbb{Q}(\beta)$.

   (a) Prove that $[F : \mathbb{Q}] = 5$.

   (b) Explain how the map 'multiplication by β' induces a $\mathbb{Q}$-linear transformation ϕ on F and a $\mathbb{Q}[T]$-module structure on F. (Here $\mathbb{Q}[T]$ is the polynomial ring in one variable over $\mathbb{Q}$.)

(c) Write down the matrix of ϕ with respect to a $\mathbb{Q}$-basis for F of your choosing. Compute all of its eigenvalues, and at least one of its eigenvectors (over $\mathbb{C}$).

7. Let $K = \mathbb{Q}(\zeta_7)$, where ζ_7 is a primitive 7th root of unity.

   (a) Find an automorphism of $K/\mathbb{Q}$ of order 6, and prove that the extension $K/\mathbb{Q}$ is Galois with Galois group isomorphic to $\mathbb{Z}/6\mathbb{Z}$.

   (b) Prove that K contains a unique cubic subfield L (i.e., with $\mathbb{Q} \subseteq L \subseteq K$ and $[L : \mathbb{Q}] = 3$).

   (c) Describe L explicitly, either by writing $L = \mathbb{Q}(\alpha)$ for some $\alpha \in K$, or by writing $L = \mathbb{Q}[x]/(f)$ for some cubic polynomial $f \in \mathbb{Q}[x]$.