Qualifying Exam in Algebra – University of South Carolina, Spring 2019

Answer all questions, closed book/notes. **GOOD LUCK!**

1. (6 points each) Let $p > q$ be primes, let G be a group of order pq, let N be a p-Sylow subgroup of G, and K be a q-Sylow subgroup of G.

   (a) Prove that N is normal in G.

   (b) Prove that, if K is also normal in G, then $G \cong \mathbb{Z}/pq\mathbb{Z}$.

   (c) Prove that, if $p = 5$ and $q = 3$, then K must be normal in G. Conclude that any group of order 15 is cyclic.

   (d) Construct a group of order 21 which is *not* cyclic.

2. (10 points) Let M be the ideal in $\mathbb{Z}[x]$ generated by 2 and x. Prove that M cannot be generated as a $\mathbb{Z}[x]$-module by a single element.

3. (10 points) Let k be a field and let G be the subgroup

$$G := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in k \right\}$$

   of 3×3 matrices with coefficients in k. Determine the center $Z(G)$ of G. Show that $Z(G) \cong k$ and $G/Z(G) \cong k^2$ where k is viewed as an additive group.

4. (10 points) Let p be a prime and let M be an $n \times n$ matrix with integer entries. Show that $\operatorname{tr}(M^p) \equiv \operatorname{tr}(M) \mod p$.

5. (6 points each) Let R be a commutative ring. Recall that the *radical* of an ideal I is the set

$$\sqrt{I} := \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{Z}^+\}.$$

   (a) Prove that $\sqrt{I}$ is an ideal.

   (b) Prove, for two ideals I and J, that $\sqrt{I} + \sqrt{J} \subseteq \sqrt{I + J}$.

   (c) The *nilradical* of a ring is $\sqrt{0}$, the ideal consisting of all nilpotent elements of that ring. Prove that $\sqrt{I}/I$ is the nilradical of R/I.

   (d) Do we always have $\sqrt{I} + \sqrt{J} = \sqrt{I + J}$? Prove or find a counterexample.

6. Let $K := \mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{7})$.

   (a) (6 points) Prove that $[K : \mathbb{Q}] = 8$.

   (b) (10 points) Prove that K is Galois over $\mathbb{Q}$. Determine its Galois group explicitly, and compute all the subfields of K.

   (c) (6 points) Find a primitive element α for K; that is, some $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$.