Honor Code Statement
I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.
As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.
Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ______________________________

Name (printed) : ______________________________

INSTRUCTIONS:

(1) Write your solutions on only one side of your paper.
(2) Start each new problem on a separate page.
(3) Write your name (or just your initials) and problem number on the top of each page.
(4) When finished put the problems in order and consecutively number your pages. Hand-in your exam, with this sheet of paper (sign the HONOR CODE STATEMENT) on top.
(5) You have 3 hours for this exam but you may take 4 hours.
(6) Questions 1-8 are each worth 10 points. Question 9 is worth 20 points.

Notation. $\mathbb{N} := \{1, 2, 3, \ldots\}$ (resp. $\mathbb{R}$, $\mathbb{C}$) denotes the set of natural (resp. real, complex) numbers.

1. Using the Residue Theorem, compute

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx \ . \tag{1.1}$$

2. Let f and g be analytic and nonzero-valued on the open disk $B_1(0) := \{z \in \mathbb{C}\colon\ |z| < 1\}$ and

$$\frac{f'\left(\frac{1}{n}\right)}{f\left(\frac{1}{n}\right)} = \frac{g'\left(\frac{1}{n}\right)}{g\left(\frac{1}{n}\right)} \qquad \text{for each } n \in \mathbb{N} \setminus \{1\} \ . \tag{2.1}$$

Show that f is a constant multiple of g on $B_1(0)$
i.e., show that there exists $k \in \mathbb{C} \setminus \{0\}$ such that, for each $z \in B_1(0)$, $f(z) = k\, g(z)$.

3. Let A and B be nonempty subsets of a metric space (X, ρ).
Define the *distance* $d(A, B)$ *between* A *and* B by

$$d(A, B) = \inf\{\rho(a, b) : a \in A, b \in B\} \ . \tag{3.1}$$

Show that if A is compact and B is closed, then $d(A, B) = 0$ if and only if $A \cap B \neq \emptyset$.

4. Let $f\colon X \to Y$ where (X, d_X) and (Y, d_Y) are nonempty metric spaces.
Show that the following are equivalent.

(1) For each *open* subset V in Y, one has $f^{-1}(V)$ is open in X.
(2) For each subset A of X, one has $f\left(\overline{A}\right) \subset \overline{f(A)}$.

If you use any *characterization of continuity* that is equivalent to the *definition of continuity* (i.e., inverse image of each open set is open), then you must also show that the used *characterization of continuity* is indeed equivalent to the *definition of continuity*. Similarly, there are several equivalent *formulations* of the definition of the closure of a set; be sure to mention which closure formulation you are using when you use it (but you do not need to show the various formulations of closure are equivalent).

5. Let A and B be subsets of a separable metric space (D, d).

(1) Define what it means for B to be separable.
(2) Show that A is separable.

Notation: $L_p((\Omega,\Sigma,\mu);\mathbb{R})$, or just L_p if confusion seems unlikely, denotes the space of equivalence classes of Σ-measurable functions $f\colon\Omega\to\mathbb{R}$ with finite $\|\cdot\|_p$-norm where $1\le p\le\infty$ and (Ω,Σ,μ) is a measure space.

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6. Let (Ω,Σ,μ) be a nonnegative finite measure space.
Let $f\colon\Omega\to\mathbb{R}$ be an μ-essentially bounded Σ-measurable function
(for such an f, recall $\|f\|_\infty := \inf\{M\ge 0\colon \mu([|f|>M])=0\}$ where $[|f|>M] := \{\omega\in\Omega\colon |f(\omega)|>M\}$).
Show that

$$\lim_{\substack{p\to\infty\\ p\in[1,\infty)}} \|f\|_p = \|f\|_\infty . \tag{6.1}$$

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7. Let a Lebesgue measurable function $f\colon [0,\infty)\to\mathbb{R}$ and $c\in\mathbb{R}$ satisfy

(1) f is Lebesgue integrable over each subinterval I of $[0,\infty)$ with $\mu(I)<\infty$

(2) $\lim_{t\to\infty} f(t)=c$.

Show that

$$\lim_{a\to\infty}\frac{1}{a}\int_{[0,a]} f\,d\mu = c . \tag{7.1}$$

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8. Let (Ω,Σ,μ) be a nonnegative finite measure space.
Let $f\in L_1((\Omega,\Sigma,\mu);\mathbb{R})$ and the sequence $\{f_n\}_{n\in\mathbb{N}}$ from L_1 satisfy

(a) $\lim_{n\to\infty} f_n = f$ μ-almost everywhere

(b) $\lim_{n\to\infty} \|f_n\|_1 = \|f\|_1$.

Show that

(1) $\lim_{n\to\infty} \int_E |f_n|\,d\mu = \int_E |f|\,d\mu$ for each $E\in\Sigma$

(2) $\lim_{n\to\infty} \|f-f_n\|_1 = 0$.

Remarks: You may use, without proving, Egoroff's Theorem provided you state Egoroff's Theorem as well as define each involved mode of converges.

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9. State whether the statement is true or false (1pt). Then either prove or give a counterexample (3pt).

9.a. For the f in (and using notation from) this exam's problem 8,
for each $\epsilon>0$ there is a $\delta>0$ such that if $E\in\Sigma$ and $\mu(E)<\delta$ then $\int_E |f|\,d\mu<\epsilon$.

9.b. The statement obtained by, in this exam's problem 6, omitting the word finite.

9.c. The statement obtained by, in this exam's problem 3, replacing A is compact with A is closed.

9.d. Let (Ω,Σ,μ) be a nonnegative measure space and $f,f_n\colon\Omega\to\mathbb{R}$ for each $n\in\mathbb{N}$.
If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of Σ-measurable functions converging μ-almost everywhere to f,
then f is Σ-measurable.

9.e. Let G be an open and connected subset of $\mathbb{C}$.
If $f,g\colon G\to\mathbb{C}$ are analytic on G and $f(z)\,g(z)=0$ for each $z\in G$,
then $f(z)=0$ for each $z\in G$ or $g(z)=0$ for each $z\in G$.